Hybrid Contingent Claims Models: A Practical Approach to Modeling Default Risk

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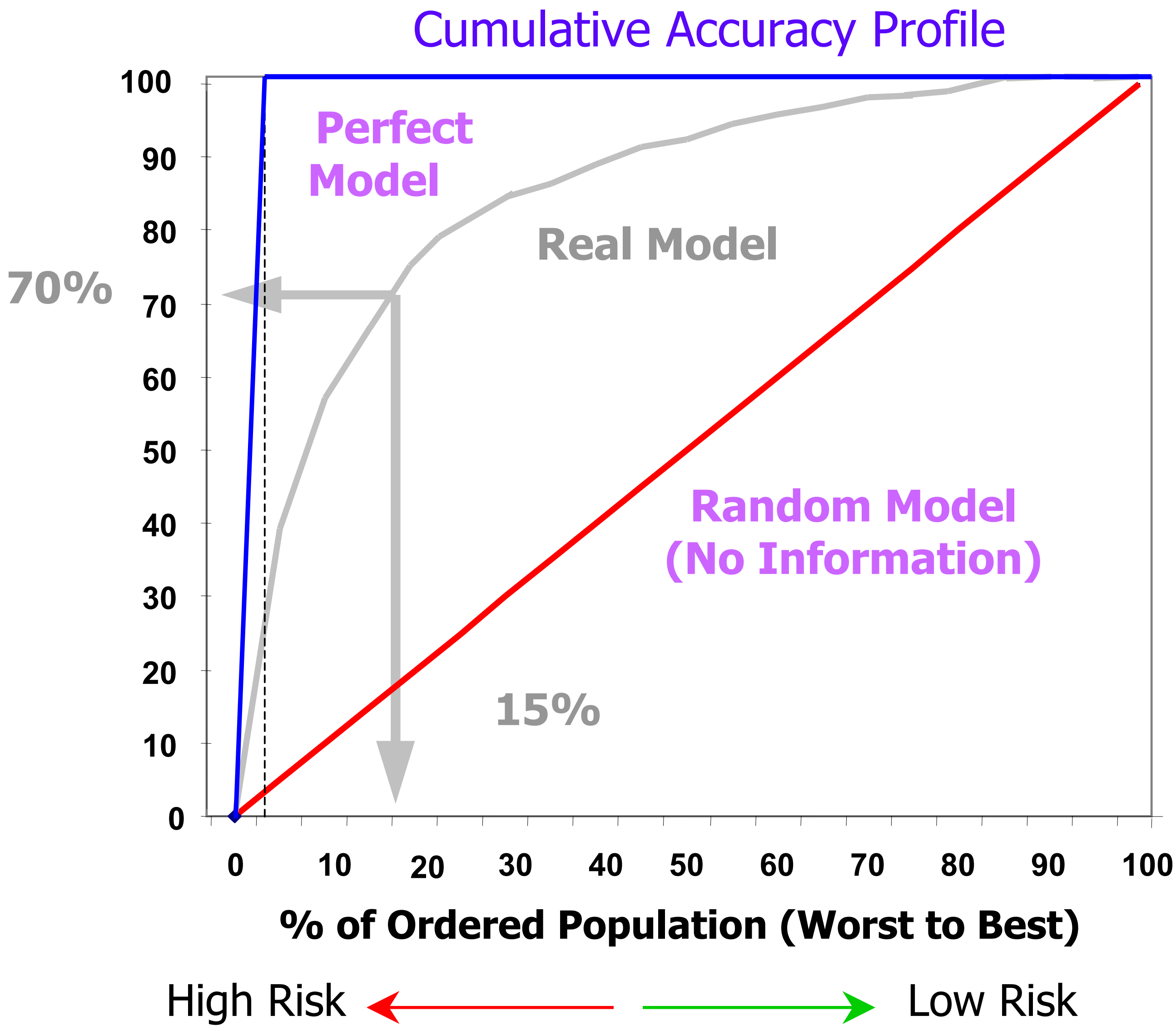
Basic Points to Make Today

- Non-market information such as financial statement information **improves** the performance of Merton models of default risk

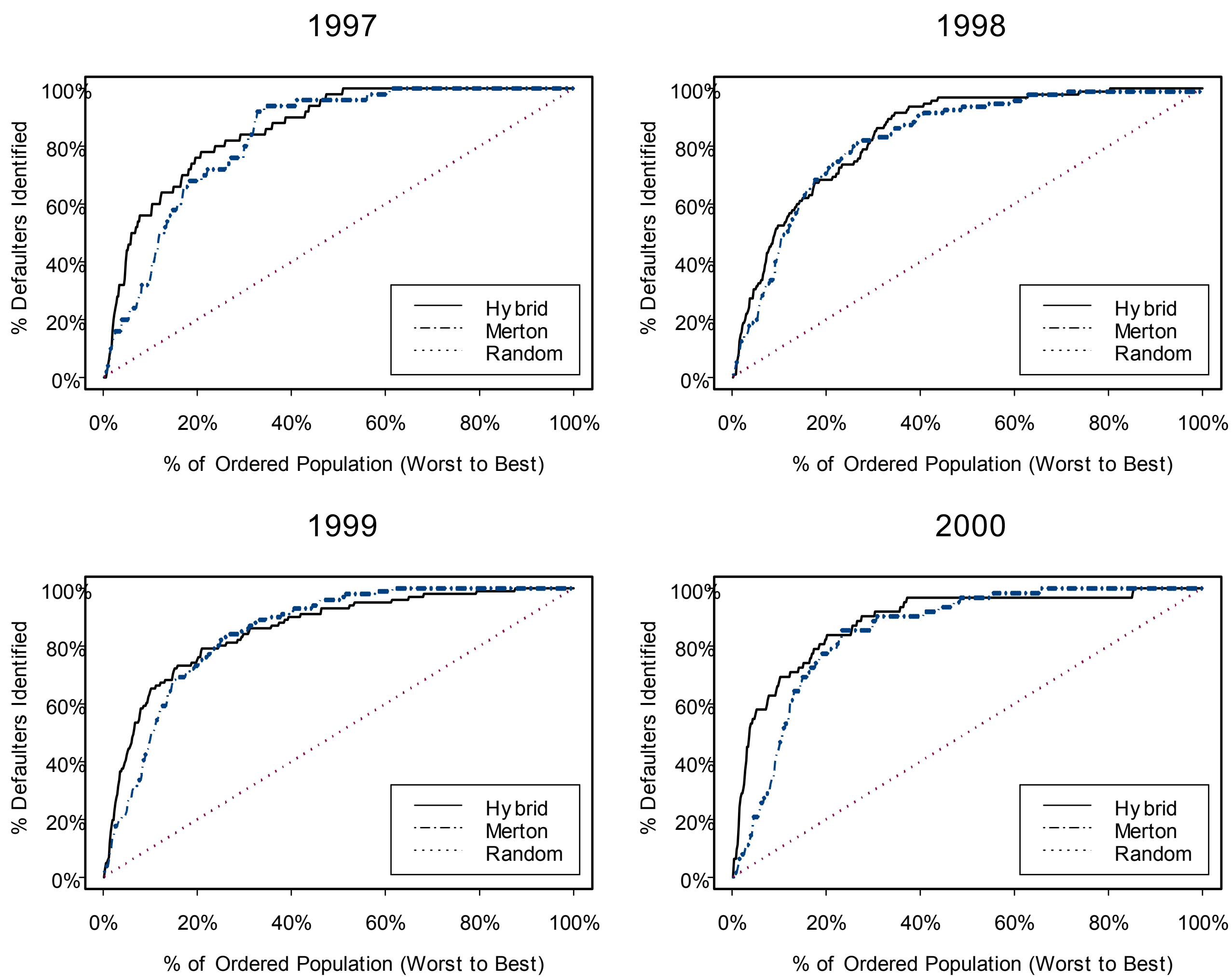
- Inclusion of market-specific uncertainty and recognition of different market dynamics for equity and debt can lead to better default prediction and pricing models

- These issues reflect some of the limitations of the **equity-as-option analogy** implied from the Merton framework

Model Performance



Model Performance in Practice: US firms



High Risk ← → **Low Risk**

Too Many Sell Signals?

- Smell Test: The sum of all 1-year default probability estimates at the beginning of the year must be similar to the number of defaults at the end of the year.

2001 Estimated Defaults

Model Type	Number of Firms	Estimated Defaults (12/2000)	Estimated All Corporate Default Rate
Merton	6,988	543	7.8%
Hybrid	6,162	198	3.2%
Hybrid Adjusted	6,162*6.9/6.1	225	3.2%

In 2001 there were **255** defaults/bankruptcies for publicly traded US companies (source www.kmv.com). Moody's provided a similar number, and an all-corporate default rate of **3.7%** (source www.moodysrms.com)

2002 Estimated Defaults

Model Type	Number of Firms	Estimated Defaults (12/2001)
Merton	6,785	527
Hybrid	6,943	274

Why Might Data Driven Default Prediction Models Match or Outperform Merton-type Models?

- Measurement Issues
  - Implied asset volatility
  - Implied market value of assets
- Debt Market Dynamics and Arbitrage
  - Liquidity limitations
  - Arbitrage opportunities alleged to exist
  - Hedging activity not observed
  - Dynamics of debt and equity markets are **very different**
- Debt Capacity
  - Inadequate description of the capacity to borrow and/or roll over debt
  - Uncertainty in future liabilities

What to Do About These Problems?

- Measurement Issues
  - Can adjust the model specification
- Debt Market Dynamics and Arbitrage
  - Can adopt a more realistic framework, accounting for multiple sources of uncertainty
  - Can include additional variables correlated with credit quality, but uncorrelated with the errors generated by the pure Merton Model
- Debt Capacity
  - Can include variables closely related to borrowing capacity

Consider Two Related Issues

- Sources of uncertainty

- Is there only a single source of uncertainty (the stochastic value of underlying assets) affecting both bond and stock markets? Is this important?

- Mechanics of Arbitrage Elimination

- Can we expect equity and assets to exhibit the same type of trading dynamics we find in stock/options markets?

Where is the Uncertainty of the Market Participants?

- The Merton framework requires market participants to **completely agree on how** to price equity and debt at each point in time even when they are uncertain about the value of the underlying assets and future liabilities.
  - Equity and debt are deterministic functions of the state variables (firm's assets, etc;)

- Absent from such scenario is the fact that, in general, participants **do not agree** on their valuation methodology:
  - Value of the firm's assets: physical vs. intellectual.
  - Volatility estimates
  - Interest rate estimates
  - Holding period
  - Claims in bankruptcy

The "Equity-as-Option" Analogy

- Initially the firm is owned outright by the equity holders.
- Equity owners, by issuing debt, "sell" the assets minus a call option on those assets to bondholders.
- At bond maturity, equity holders either

  A) exercise the call, paying the strike price (face value of debt) and reclaiming ownership of the assets,

  or

  B) fail to exercise the call (default on the debt) if the asset value is lower than the face value of the debt (option is out of the money).

The "Equity-as-Option" Analogy, cont.

- Why would Equity holders want to convert their outright ownership into a call option on the firm's assets?

- Leverage
- Limited liability
- Or, to hedge their exposure to a sudden fall in asset value, the same way stock holders use call options to hedge their stock portfolios.

Under these conditions, we would expect to see robust markets to support the need of owners to hedge their positions, and in combination with speculators, to price the risk of the outcomes being hedged against.

Is There Any Hedging Occurring in the "Equity-as-Option" Analogy?

Here we are not talking about the dynamic hedging associated with the N.A. condition, but rather with the market dynamics likely to emerge, given the motivation of the actual market participants.

- Equity holders receive cash now, and can walk away from the assets if there is a sudden fall in asset value, *but*

- for the hedge to be effective, managers could not invest the proceeds from bond sales in additional assets, *and*

- equity holders would have to be able to protect the cash raised from creditors, post default.

Both are False!!

Illustrative Examples:

Market Uncertainty

Debt Capacity

Fundamental Assumptions of the Merton Model

Assumption 1: The price of equity and debt are deterministic functions of:

the firm's assets value A (or other state variables)

$$Equity = E(A,t) \quad Debt = D(A,t) \qquad A = E + D$$

Assumption 2: An ideal self-financing hedge portfolio composed of the firms assets, ideal debt (or equity), and riskless bonds can be constructed. Ideal debt is assumed to be marketable debt.

$$P\,(\textbf{assets, marketable debt}) = P(A,D,t)$$

Assumption 3: The ideal portfolio can be hedged at no cost at each point in time to remove all stochastic variations.

$$\partial_A P\, dA = 0 \Rightarrow \quad dP = P\, R\, dt \qquad \text{where } R = 0$$

(no arbitrage condition)

Limitations

- The debt and equity markets uncertainty **is not embedded** into the Merton model. Assumption (1) establishes this restriction.
  - The model does not admit any **uncertainty** of the market participants ***on how*** to price either equity or debt.
    - This makes the Merton model equivalent to the Black-Scholes model.
  - Importantly, **only** the (unobservable) assets' uncertainty is represented in the Merton model.
    - This allows to hedge the ideal portfolio and impose the no-arbitrage condition
    - No uncertainty of the level of liabilities and option to renegotiate or refinance

A Re-evaluation of the Merton Framework Should Include Both:

- A recognition that **different** markets for derivatives and their underlying securities implies separate sources of **intrinsic uncertainty**.
  - Market equity and debt may not be **deterministic** functions of the firm's assets and time.
    - E or D = Function(A,t) + market noise

- A careful **review** of the no arbitrage condition for market equity and debt.
  - What type of equilibrium condition can be established if a riskless hedge portfolio **cannot be** maintained due to liquidity problems, changing liabilities and unobservable assets?
    - E + D = A + noise caused by market participants

Market Uncertainty and Implied Assets and Volatility

Assume that the Merton model

$$F(A_I, \sigma_I) = A_I \; N(d_1) - e^{-r(T-t)} \; D_0 \; N(d_2) \qquad (1)$$

with implied assets and volatility is a good approximation to the observed market equity $E(\mathrm{A}, \sigma)$ with the true assets and volatility.

We need to determine the "implied" firm's assets and volatility A_I and σ_I from equity E and equity volatility σ_E, which are functions of the true values A and σ.

$$E \approx F(A, \sigma) + \partial_A F \; (A_I - A) + \partial_\sigma F \; (\sigma_I - \sigma) \qquad (2)$$

$$\sigma_E^2 \approx \Sigma^2(A, \sigma) + \partial_A \Sigma^2 \; (A_I - A) + \partial_\sigma \Sigma^2 \; (\sigma_I - \sigma) \qquad (3)$$

Here

$$\Sigma^2(A_I, \sigma_I) = \left(\frac{A}{F} \frac{\partial F}{\partial A} \right)^2 \sigma^2 \qquad (4)$$

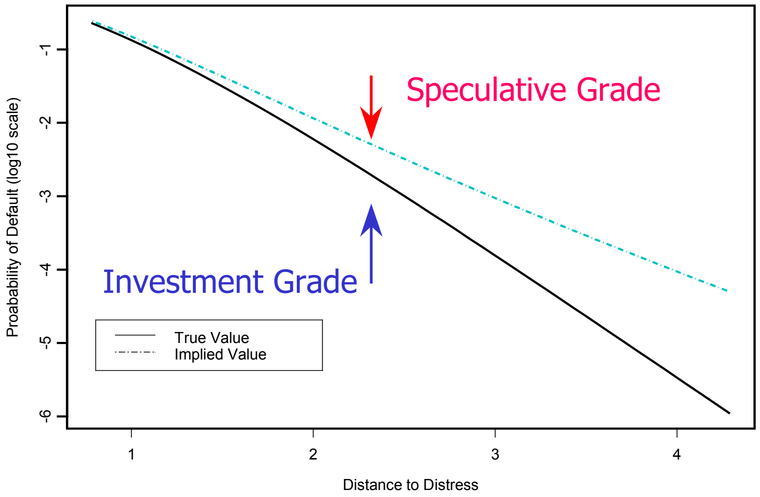
Example of Model Bias: Probability of Default

To illustrate, let's assume that the "actual" market equity is given by the Merton model with noise X and a risk premium due to uncertainty

$$E = \mathrm{E}^{X}[e^{X} F(A, \sigma, \rho)] \quad \rho \geq r, \;\; X \sim \text{noise} \tag{5}$$

Using equations (1)-(5) we determine the implied assets and volatility and calculate the actual and implied probabilities of default

Probability of Default



High Risk ← → Low Risk

Debt Capacity

- If the firm has **borrowing capacity**, it can reduce the amount due by rolling over its debt, reducing its probability of default.
- There are two popular assumptions imposed on the borrowing capacity of the firm:

  (a) **Negligible borrowing capacity**
    - single balloon payment models
      - European option (Merton (1973,1974))
      - First passage time (Longstaff-Schwatz (1995))

  (b) **Infinite borrowing capacity (costless)**
    - continuous roll-over of debt at no cost
      - perpetual options model (Leland (1999), Kealhofer (1999) and others).

Debt Capacity and Accounting Variables

- Consider that the assumptions of the Merton model are valid.
- At maturity T the firm can refinance a fraction F of the par value of the debt D at an additional interest expense I that is paid at the maturity of the first obligation as a refinancing fee.
- The net effect is to reduce the level of the firm's original obligations to $(1-F)D + I$.
- How much can be rolled over and how much needs to be charged in interest depends on the lender's policy and the expectation of the borrower's ability to repay at future date.
  - This is a compounded option problem where the first default point depends on the second default point and vice versa.
  - The lender's policy for F and I is a function of profitability (P), liquidity (L), capital structure (C), etc. (It can also depend on agency ratings.)

Adding Critical Information

- The refinanced fraction F has the effect of adjusting the leverage in the Merton model. Therefore, the probability of default PD_M at time T on the first obligation is:

$$PD_M(z, T-t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx \tag{7}$$

$$\begin{aligned} y(z, T-t) &= \frac{1}{\sigma\sqrt{T-t}} \log\left(\frac{Ae^{\mu(T-t)}}{D^*}\right) \\ &= \frac{1}{\sigma\sqrt{T-t}} \left[\log\left(\frac{Ae^{\mu(T-t)}}{D}\right) - \log\left(1 - F(P, L, C, ...) + \frac{I(P, L, C, ..)}{D}\right)\right] \end{aligned} \tag{8}$$

- Here $z = [D(1-F)+I]/A$ is the **effective leverage** that accounts for the borrowing and negotiation capacity of the firm.
- Equation (7) is a probit model in which **financial ratios** are key regressors (simple example of a hybrid model).

Enhancing the Concept of Default Point

- In the Merton framework once the assets of the firm reach a critical value (or default point), default is inevitable

$$p(\text{default ocurred at } T \mid z_T) = \begin{cases} 1 & z_T \geq 1 \\ 0 & z_T < 1 \end{cases} \qquad (9)$$

- We can relax equation (9) by assuming that the event of default becomes **more likely** but not necessarily **inevitable**.
- We can also allow default to occur if $z < 1$ but with lower probability to account for cases where otherwise solvent firms file for bankruptcy due to:
  - future legal liabilities
  - severe industry downturns
  - unanticipated catastrophic events

The Impact of Unanticipated Events

- There are several reasons why we may want to relax equation (9)
  - The firm's assets are generally unobservable
  - Some liabilities are non-tradable or illiquid
  - the firm's leverage does not define the default point under every possible scenario
- Thus, there is no reason to believe that the idealized condition (9) holds exactly.
- Allowing the conditional default probability at time *T* to be a function $g(z_T)$ of the state variable z_T, the probability of default is:

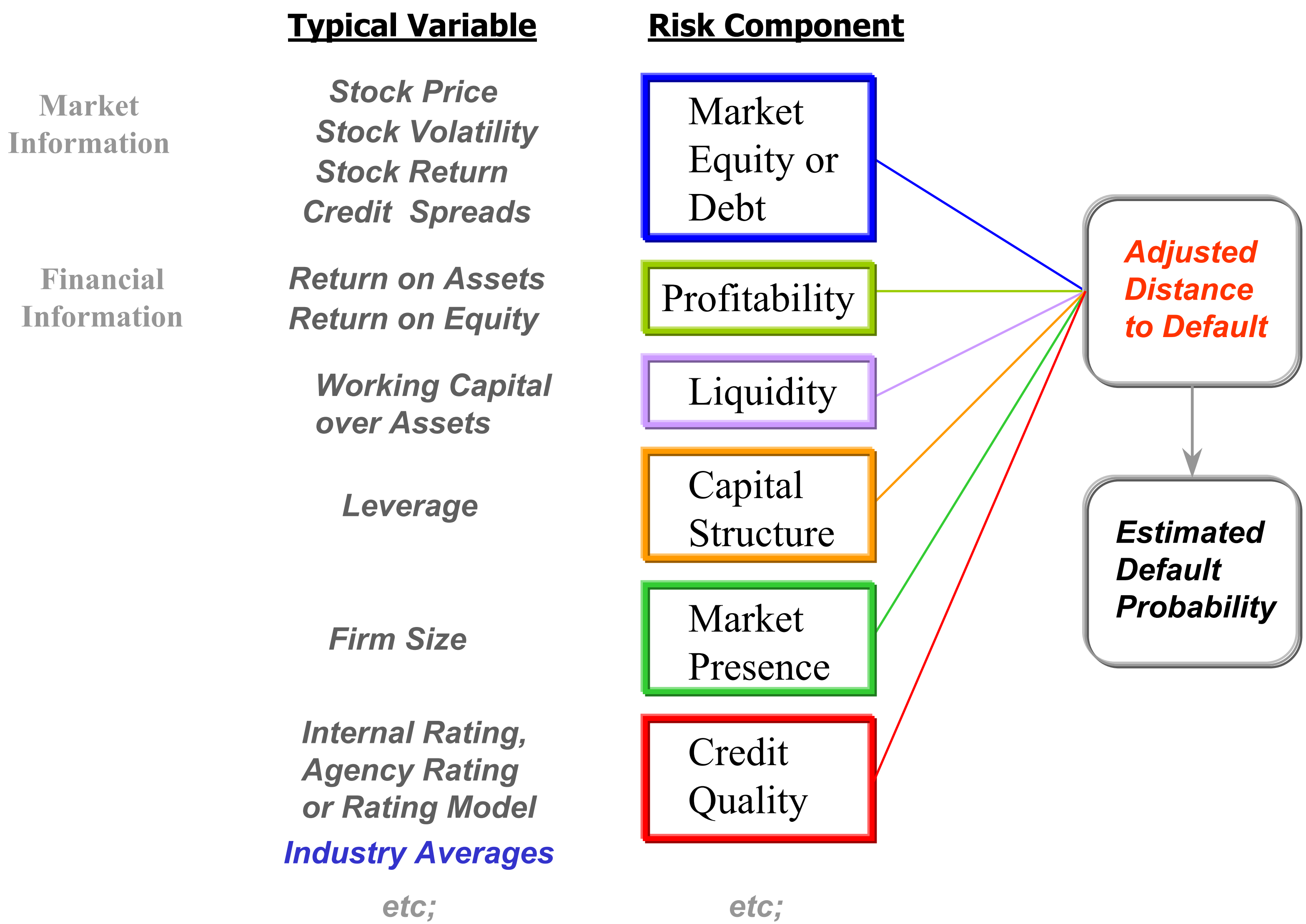
$$PD(z,T-t) = \int_0^\infty g(z_T)\, f(z_T,z)\; dz_T$$

$$= PD_M(z,T-t) - \int_1^\infty \left(1-g(z_T)\right) f(z_T,z)\, dz_T + \int_0^1 g(z_T)\, f(z_T,z)\, dz_T \qquad (10)$$

Here f(*z*) is the log-normal distribution of the effective leverage *z*.

The function g(*z*) can depend on industry, business environment and regulatory factors.

The Structure of Hybrid Models in Practice



Conclusions

- A recognition that independent sources of uncertainty exists may lead to more realistic default risk and pricing models.

- Non-market information such as financial statement information **improves** the performance of Merton models for default prediction both
    - in theory
    - in practice